Univalent Mathematics: Theory and Implementation

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ICMS, July 15, 2020

International Congress on Mathematical Software (ICMS) 2020

Organized by TU Braunschweig, 13-17 July 2020:

`http://www.iaa.tu-bs.de/AppliedAlgebra/ICMS2020/ICMS2020.html`

ICMS is a bi-annual congress that gathers the mathematicians, scientists and programmers who are interested in the development of **mathematical software**.

This year it has 14 sessions, 5 software demos and 3 plenary speakers:

- Erika Ábrahám (RWTH Aachen): Solving Real-Algebraic Formulas with SMT-RAT
- Alan Edelman (MIT): Julia – The Power of Language
- Victor Shoup (Courant Institute): NTL: a Library for Doing Number Theory

Practical information

Plenary talks will be held live over Zoom; links will be emailed to registered participants.

There will also be social events, including a cooking competition for the conference dinner.

Sessions will consist of pre-recorded videos for the talks:

`https://djauu93qmtiza.cloudfront.net/`

Our session meets virtually on **Wednesday, July 15** for 10-minute discussions of each talk.

We also have a channel (`#session-i`) on the ICMS 2020 Slack. (Check your email.)

Type theory (a very rough historical outline)

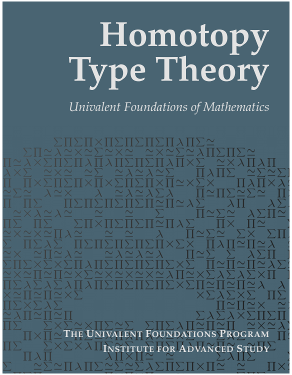
Dependent type theory is a constructive logic used in many proof assistants.

- 1972: Martin-Löf invents modern type theory (MLTT).
- 1988: Coquand invents calculus of constructions.
- 1989: `Coq`.
- 1990: `ALF` (predecessor of `Agda`).
- 2005–: Ideas behind HoTT/UF first appear (Voevodsky, Awodey–Warren, others).
- 2007: `Agda` 2.
- 2012–2013: IAS Special Year on Univalent Foundations of Mathematics.
- 2013: `Lean`.

Formalized mathematics in HoTT/UF

Develop **univalent mathematics/synthetic homotopy theory** using MLTT+axioms.

- 2010: Voevodsky's Foundations library in `Coq`.
- 2011–: HoTT `Coq` library.
- 2011–: HoTT `Agda` library.
- 2012–2013: HoTT Book.
- 2014–: UniMath library in `Coq`.
- 2015–: HoTT `Lean` library.



ICMS 2016

Session on **Univalent Foundations and Proof Assistants** organized by Vladimir Voevodsky:

- Anand: Exploiting uniformity in substitution: the Nuprl term model
- Ahrens: Inductive sets in UniMath
- Altenkirch: A Cubical Type Theory
- Bezem: A taxonomy of mathematical mistakes
- Brunerie: Custom definitional equalities in Agda
- Bickford: A model of Cubical Type Theory in Nuprl
- Gross: The HoTT/HoTT Library in Coq: Designing for Speed
- Lelay: A construction of real numbers in UniMath
- Mörtberg: Cubical Type Theory
- Rahli: Exercising Nuprl's Open-Endedness
- Sozeau: Coq for Univalent Foundations
- van Doorn: The Lean HoTT library
- von Raumer: Formalizing Double Groupoids and Cross Modules in the Lean Theorem Prover

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https://icms2016unimath.github.io/mainpage

What's happened since 2016?

Several of these libraries are still actively developed.

Many new proof assistants inspired by lessons learned:

- Agda-$\flat$
- Andromeda
- Arend
- Cubical Agda
- redtt

We have talks related to all these topics this year!

The role of computation in type theory

Type theory is a constructive logic—implementations are essentially functional PLs with very expressive type systems, using computation to determine many equations automatically.

Axioms are uninterpreted constants which disrupt computation—a PL where we don't know how to run all programs. **Fewer automatic equations.**

Can we develop univalence + HITs in a (more) computationally-friendly way?

Making HoTT/UF compute

This motivated **cubical type theories**, leading to **cubical proof assistants**.

$\implies$ `Cubical Agda` and `redtt`
$\implies$ `Arend` (somewhere in between HoTT/UF and cubical systems)

See the talks by Isaev, Pujet, Vezzosi, and Favonia!

Extending HoTT

Many concepts can't be expressed without further extensions to HoTT/UF.

Topological (cf homotopical) aspects of spaces can be expressed in cohesive type theory using **modalities**. Also helpful when formalizing cubical models (e.g., Orton–Pitts).

$\Longrightarrow$ `Agda-♭`

See the talk by Gratzer!

Extending HoTT

HoTT/UF is a language for ∞-groupoids. What about ∞-**categories**?
(Extends cubical type theory with an asymmetric notion of "equality.")

See the talk by Weaver!

What about **parameterized spectra**? (Requires aspects of linear/bunched logic.)

See the talk by Riley!

Equality in type theory

One of the hardest parts of justifying or implementing a new type theory is its **equality**.

Can we systematize extending type theories with new constructs and **new equalities**, besides those that fit into existing rule schemas? $\Longrightarrow$ extension to `Agda`

See the talk by Cockx!

Can we build a proof assistant that supports user-defined theories, while also having good equality-checking algorithms right out of the box? $\Longrightarrow$ `Andromeda`

See the talk (and paper) by Petković!

Our session: https://univalent-math.github.io/

Meet virtually on **Wednesday**, **July 15**, for discussions (times in CEST/UTC+2).

Part 1	Chair: Anders Mörtberg	
14:00-14:10	Angiuli, Mörtberg	Univalent Mathematics: Theory and Implementation
14:10-14:20	Isaev	Arend proof assistant
14:20-14:30	Petković	Andromeda 2 - your type theory à la carte

Part 2	Chair: Carlo Angiuli	
15:50-16:00	Pujet	Cubical Coq using Intensional Presheaves
16:00-16:10	Vezzosi	Indexed Families in Cubical Agda
16:10-16:20	Favonia	Nullable Compositions

Part 3	Chair: Anders Mörtberg	
16:30-16:40	Weaver	A constructive model of directed univalence in bicubical sets
16:40-16:50	Riley	A Type Theory for Parameterised Spectra
16:50-17:00	Cockx	Rewriting Type Theory
17:00-17:10	Gratzer	Multimodal Dependent Type Theory